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TRIPLE SEQUENCE SPACES OF INTUITIONISTIC ROUGH $I-$ CONVERGENCE DEFINED BY COMPACT BERSTEIN OPERATOR

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Abstract. This paper is to introduce the triple sequence spaces of intuitionistic rough I-convergent of $B_{\Lambda^3(\mu,\gamma)}(f,x,T)$ and $B_{\chi^3(\mu,\gamma)}(f,x,T)$ are defined by compact Bernstein operator and study the topology general properties.

Keywords: Triple sequences, rough convergence, closed and convex, cluster points and rough limit points, compact, Bernstein polynomials, Intuitionistic I-convergence.
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1. Introduction

The idea of rough convergence was first introduced by Phu [10-12] in finite dimensional normed spaces. He showed that the set LIM_x^r is bounded, closed and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of LIM_x^r on the roughness of degree r.

Aytar [1] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [2] studied that the $r-$ limit set of the sequence is equal to intersection of these sets and that $r-$ core of the sequence is equal to the union of these sets. D$\ddot{u}$ndar and Cakan [8] investigated of rough ideal convergence and defined the set of rough ideal limit points of a sequence The notion of $I-$ convergence of a triple sequence which is based on the structure of the ideal I of subsets of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, where $\mathbb{N}$ is the set of all natural numbers, is a natural generalization of the notion of convergence and statistical convergence.

In this paper we investigate some basic properties of rough $I-$ convergence of a

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triple sequence in three dimensional matrix spaces which are not earlier. We study the set of all rough $I-$ limits of a triple sequence and also the relation between analytic ness and rough $I-$ convergence of a triple sequence.

Let K be a subset of the set of positive integers $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and let us denote the set $K_{ik\ell}=\{(m,n,k)\in K: m\leq i, n\leq j, k\leq \ell\}$. Then the natural density of K is given by

$$\delta(K)=lim_{i,j,\ell\to\infty}\frac{|K_{ij\ell}|}{ij\ell},$$

where $|K_{ij\ell}|$ denotes the number of elements in $K_{ij\ell}$. Clearly, a finite subset has natural density zero, and we have $\delta(K^c)=1-\delta(K)$ where $K^c=\mathbb{N}\backslash K$ is the complement of K. If $K_1\subseteq K_2$, then $\delta(K_1)\leq\delta(K_2)$.

The Bernstein operator of order (r,s,t) is given by
$B_{rst}(f,x)=\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}f\left(\frac{mnk}{rst}\right)\binom{r}{m}\binom{s}{n}\binom{t}{k}x^{m+n+k}(1-x)^{(m-r)+(n-s)+(k-t)}$
where f is a continuous (real or complex valued) function defined on $[0,1]$.

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers , χ_A- the characteristic function of $A\subset\mathbb{N},\mathbb{R}$ the set of all real numbers. A subset A of $\mathbb{N}$ is said to have asymptotic density $d(A)$ if

$$d(A)=lim_{i,j,\ell\to\infty}\frac{1}{ij\ell}\sum_{m=1}^{i}\sum_{n=1}^{j}\sum_{k=1}^{\ell}\chi_A(K).$$

Throughout the paper, $\mathbb{R}$ denotes the real of three dimensional space with metric (X,d). Consider a triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ such that $(B_{mnk}(f,x))\in\mathbb{R}, m,n,k\in\mathbb{N}$.

Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ is said to be statistically convergent to $0\in\mathbb{R}$, written as $st-lim\ x=0$, provided that the set

$$K_\epsilon:=\{(m,n,k)\in\mathbb{N}^3: |B_{mnk}(f,x)-f(x)|\geq\epsilon\}$$

has natural density zero for any $\epsilon>0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein polynomials. i.e., $\delta(K_\epsilon)=0$. That is,

$$lim_{r,s,t\to\infty}\frac{1}{rst}|\{m\leq r, n\leq s, k\leq t: |B_{mnk}(f,x)-(f,x)|\geq\epsilon\}|=0.$$

In this case, we write $\delta-limB_{mnk}(f,x)=f(x)$ or $B_{mnk}(f,x)\to^{S_B}f(x)$.

A triple sequence of Bernstein operator of $(B_{mnk}(f,x))$ is said to be statistically convergent to $f(x)\in\mathbb{R}$, written as $st-lim\ B_{mnk}(f,x)=f(x)$, provided that the set

$$\{(m,n,k)\in\mathbb{N}^3: |B_{mnk}(f,x),f(x)|\geq\epsilon\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$.

If a triple sequence of Bernstein polynomial is statistically convergent, then for every $\epsilon > 0$, infinitely many terms of the sequence may remain outside the $\epsilon-$ neighbourhood of the statistical limit, provided that the natural density of the set consisting of the indices of these terms is zero. This is an important property that distinguishes statistical convergence from ordinary convergence. Because the natural density of a finite set is zero, we can say that every ordinary convergent sequence is statistically convergent.

If a triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ satisfies some property P for all m, n, k except a set of natural density zero, then we say that the triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ satisfies P for almost all (m, n, k) and we abbreviate this by a.a. (m, n, k).

Let $\left(B_{m_i n_j k_\ell}(f,x)\right)$ be a sub sequence of $(B_{mnk})(f,x)$. If the natural density of the set $K = \{(m_i, n_j, k_\ell) \in \mathbb{N}^3 : (i, j, \ell) \in \mathbb{N}^3\}$ is different from zero, then $\left(B_{m_i n_j k_\ell}(f,x)\right)$ is called a non thin sub sequence of a triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$.

$c \in \mathbb{R}$ is called a statistical cluster point of a triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ provided that the natural density of the set

$$\{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}(f,x) - c| < \epsilon\}$$

is different from zero for every $\epsilon > 0$. We denote the set of all statistical cluster points of the sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ by Γ_x.

A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ is said to be statistically analytic if there exists a positive number M such that

$$\delta\left(\left\{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}(f,x)|^{1/m+n+k} \geq M\right\}\right) = 0$$

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by *Sahiner et al. [13,14]*, *Esi et al. [3-6]*, *Datta et al.*

[7],*Subramanian et al. [15]*, *Debnath et al. [8]* and many others.

A triple sequence $x = (x_{mnk})$ is said to be triple analytic if

$$sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The space of all triple analytic sequences are usually denoted by Λ^3.

A triple sequence $x = (x_{mnk})$ is said to be triple chi if

$$((m+n+k)|x_{mnk}|)^{\frac{1}{m+n+k}} \to 0, \text{ as } m,n,k \to \infty.$$

The space of all triple chi sequences are usually denoted by χ^3.

2. Definitions and Preliminaries

Throughout the paper $\mathbb{R}^3$ denotes the real three dimensional case with the metric. Consider a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}^3; m,n,k \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. The following definition are obtained:

2.1. **Definition.** Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers is said to be statistically convergent to $f(x) \in \mathbb{R}^3$ if for any $\epsilon > 0$ we have $d(A(\epsilon)) = 0$, where

$$A(\epsilon) = \{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}(f,x) - f(x)| \geq \epsilon\}.$$

2.2. **Definition.** Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers is said to be statistically convergent to $f(x) \in \mathbb{R}^3$, written as $st - limB_{mnk}(f,x) = f(x)$, provided that the set

$$\{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}(f,x) - f(x)| \geq \epsilon\},$$

has natural density zero for every $\epsilon > 0$.

In this case, l is called the statistical limit of the sequence $B_{mnk}(f,x)$.

2.3. **Definition.** Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers of a metric space $(X, |.,.|)$ and r be a non-negative real number is said to be $r-$convergent to $f(x) \in X$, denoted by $B_{mnk}(f,x) \to^r f(x)$, if for any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that for all $m,n,k \geq N_\epsilon$ we have

$$|B_{mnk}(f,x) - f(x)| < r + \epsilon$$

In this case $f(x)$ is called an $r-$ limit of $B_{mnk}(f,x)$.

2.4. **Remark.** We consider $r-$ limit set $B_{mnk}(f,x)$ which is denoted by $LIM^r_{B_{mnk}(f,x)}$ and is defined by

$$LIM^r_{B_{mnk}(f,x)} = \{f(x) \in X : B_{mnk}(f,x) \to^r f(x)\}.$$

2.5. **Definition.** Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers is said to be $r-$ convergent if $LIM^r_{B_{mnk}(f,x)} \neq \phi$ and r is called a rough convergence degree of $B_{mnk}(f,x)$. If $r=0$ then it is ordinary convergence of triple sequence of Bernstein polynomials.

2.6. **Definition.** Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers of a metric space $(X, |.,.|)$ and r be a non-negative real number is said to be $r-$ statistically convergent to $f(x)$, denoted by $B_{mnk}(f,x) \to^{r-st_3} f(x)$, if for any $\epsilon > 0$ we have $d(A(\epsilon)) = 0$, where

$$A(\epsilon) = \{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |B_{mnk}(f,x) - f(x)| \geq r + \epsilon\}.$$

In this case $f(x)$ is called $r-$ statistical limit of $B_{mnk}(f,x)$. If $r=0$ then it is ordinary statistical convergent of triple sequence of Bernstein polynomials.

2.7. **Definition.** A class I of subsets of a nonempty set X is said to be an ideal in X provided
(i) $\phi \in I$
(ii) $A, B \in I$ implies $A \bigcup B \in I$.
(iii) $A \in I, B \subset A$ implies $B \in I$.
I is called a nontrivial ideal if $X \notin I$.

2.8. **Definition.** A nonempty class F of subsets of a nonempty set X is said to be a filter in X. Provided
(i) $\phi \in F$.
(ii) $A, B \in F$ implies $A \bigcap B \in F$.
(iii) $A \in F, A \subset B$ implies $B \in F$.

2.9. **Definition.** I is a non trivial ideal in X, $X \neq \phi$, then the class

$$F(I) = \{M \subset X : M = X \backslash A \; for\; some\; A \in I\}$$

is a filter on X, called the filter associated with I.

2.10. **Definition.** A non trivial ideal I in X is called admissible if $\{x\} \in I$ for each $x \in X$.

2.11. **Note.** If I is an admissible ideal, then usual convergence in X implies I convergence in X.

2.12. **Remark.** If I is an admissible ideal, then usual rough convergence implies rough $I-$ convergence.

2.13. **Definition.** Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers of a metric space $(X, |.,.|)$ and r be a non-negative real number is said to be rough ideal convergent or $rI-$ convergent to $f(x)$, denoted by $B_{mnk}(f,x) \to^{rI} f(x)$, if for any $\epsilon > 0$ we have

$$\{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |B_{mnk}(f,x) - f(x)| \geq r + \epsilon\} \in I.$$

In this case $f(x)$ is called $rI-$ limit of $B_{mnk}(f,x)$ and a triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ is called rough $I-$ convergent to $f(x)$ with r as roughness of degree. If $r = 0$ then it is ordinary $I-$ convergent.

2.14. **Note.** Generally, Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,y))$ of real numbers is not $I-$ convergent in usual sense and $|B_{mnk}(f,x) - B_{mnk}(f,y)| \leq r$ for all $(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ or

$$\{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |B_{mnk}(f,x) - B_{mnk}(f,y)| \geq r\} \in I.$$

for some $r > 0$. Then the triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ is $rI-$ convergent.

2.15. **Note.** It is clear that $rI-$ limit of $B_{mnk}(f,x)$ is not necessarily unique.

2.16. **Definition.** Consider $rI-$ limit set of $B_{mnk}(f,x)$, which is denoted by

$$I - LIM^{r}_{B_{mnk}(f,x)} = \left\{ f(x) \in X : B_{mnk}(f,x) \to^{rI} f(x) \right\},$$

then the triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ is said to be $rI-$ convergent if $I - LIM^{r}_{B_{mnk}(f,x)} \neq \phi$ and r is called a rough $I-$ convergence degree of $B_{mnk}(f,x)$.

2.17. **Definition.** Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers is said to be $I-$ analytic if there exists a positive real number M such that

$$\left\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : |B_{mnk}(f,x)|^{1/m+n+k} \geq M\right\} \in I.$$

2.18. **Definition.** A point $f(x) \in X$ is said to be an $I-$ accumulation point of a triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ in a metric space (X,d) if and only if for each $\epsilon > 0$ the set

$$\{(m,n,k) \in \mathbb{N}^3 : d(B_{mnk}(f,x), f(x)) = |B_{mnk}(f,x) - f(x)| < \epsilon\} \notin I.$$

We denote the set of all $I-$ accumulation points of $B_{mnk}(f,x)$ by $I(\Gamma_x)$.

2.19. **Definition.** Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers, the notions of ideal limit superior and ideal limit inferior are defined as follows:

$$I - limsup\, x = \left\{\begin{array}{cc} sup\, B_x, & \text{if} B_x \neq \phi, \\ -\infty, & \text{if} B_x = \phi \end{array}\right\},$$

and

$$I - liminf\; x = \left\{\begin{array}{cc} inf\; A_x, & \text{if} A_x \neq \phi, \\ +\infty, & \text{if} A_x = \phi \end{array}\right\},$$

where $A_x = \{a \in \mathbb{R} : \{(m,n,k) \in \mathbb{N}^3 : B_{mnk}(f,x) < a\} \notin I\}$ and $B_x = \{b \in \mathbb{R} : \{(m,n,k) \in \mathbb{N}^3 : B_{mnk}(f,x) > b\} \notin I\}$.

2.20. **Definition.** Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers is said to be rough $I-$ convergent if $I - LIM^r B_{mnk}(f,x) \neq \phi$. It is clear that if $I - LIM^r B_{mnk}(f,x) \neq \phi$ for a triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers, then we have $I - LIM^r B_{mnk}(f,x) = [I - limsup\, x - r, I - liminf\; x + r]$.

2.21. **Definition.** Let $(X;\mu;\nu;*;\diamond)$ be an IRMS (*intuitionistic rough metric space*). Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers is said to be $r-$ convergent to $f(x) \in X$ with respect to the intuitionistic rough metric $(\mu;\nu)$ if, for every $\epsilon > 0$ and $t > 0$, there exists $m_0, n_0, k_0 \in \mathbb{N}^3$ such that $\mu(B_{mnk}(f,x) - f(x), t) > 1-(r+\epsilon)$ and $\nu(B_{mnk}(f,x) - f(x), t) < r + \epsilon$ for all $m \geq m_0, n \geq n_0, k \geq k_0$. In this case we write $(\mu,\nu) - lim(B_{mnk}(f,x)) = f(x)$.

2.22. **Definition.** Let $(X;\mu;\nu;*;\diamond)$ be an IRMS (*intuitionistic rough metric space*). Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers is said to be $r-$ Cauchy sequence with respect to the intuitionistic rough metric $(\mu;\nu)$ if for every $\epsilon > 0$ and $t > 0$ there exists $m_0, n_0, k_0 \in \mathbb{N}^3$ such that $\mu(B_{mnk}(f,x) - f(x), t) < (r+\epsilon)$ and $\nu(B_{mnk}(f,x) - f(x), t) < r + \epsilon$ for all $m \geq m_0, n \geq n_0, k \geq k_0$.

2.23. **Definition.** Let $I \subset 2^{N^3}$ be a non-trivial ideal and $(X;\mu;\nu;*;\diamond)$ be an IRMS (*intuitionistic rough metric space*). Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers is said to be rI convergent to $f(x) \in X$ with respect to the intuitionistic rough metric (μ,ν) if for every $\epsilon > 0$ and $t > 0$, the set
$\{(m,n,k) \in \mathbb{N}^3 : \mu(B_{mnk}(f,x) - f(x), t) > 1 - (r+\epsilon) \ or \ \nu(B_{mnk}(f,x) - f(x), t) < r + \epsilon\} \in rI$.

In this case $f(x)$ is called the $rI-$ limit of the triple sequence space of Bernstein polynomials with respect to the intuitionistic rough metric (μ,ν) and we write $I_{(\mu,\nu)} - lim B_{mnk}(f,x) = f(x)$.

2.24. **Definition.** Let X and Y be two metric linear spaces. An operator $T : X \times X \times X \longrightarrow Y \times Y \times Y$ is said to be compact linear operator (or completely continuously linear operator), if
(i) T is linear,
(ii) T maps every triple analytic sequence space of Bernstein polynomials of $(B_{mnk}(f,x)) \in X$ on to a triple sequence space of Bernstein polynomials of $(T(B_{mnk}(f,x)))$ in Y which has a convergent subsequence.

The set of all compact Bernstein linear operators $C(B_{mnk}(f,X), B_{mnk}(f,Y))$ is a closed subspace of $D(B_{mnk}(f,X), B_{mnk}(f,Y))$ and $C(B_{mnk}(f,X), B_{mnk}(f,Y))$ is Banach metric space, if Y is a Banach metric space.

Now introduce the following triple chi sequence space of compact Bernstein operator:

$B^{rI}_{\chi^3(\mu,\nu)}(T) =$
$\{x \in \Lambda^3 : \{\mu(T(B_{mnk}(f,x) - f(x), t)) \leq 1 - (r+\epsilon) \ or \ \nu(T(B_{mnk}(f,x) - f(x), t)) \geq r + \epsilon\}\} \in rI$.

We also define an open ball with centre x and radius r with respect to t as follows:
$D_{B_{mnk}(f,x)}(r,t)(T) = x \in \Lambda^3 :$

$\mu\left(T\left(B_{mnk}\left(f,x\right)-T\left(B_{mnk}\left(f,y\right)\right),t\right)\right)\leq 1-(r+\epsilon)$ *or* $\nu\left(T\left(B_{mnk}\left(f,x\right)-T\left(B_{mnk}\left(f,y\right)\right),t\right)\right)\geq r+\epsilon\in rI.$

3. Main Results

3.1. **Theorem.** Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of compact Bernstein polynomials of $\left(B_{mnk}\left(f,x\right)\right)$ of real numbers, $B^{I}_{\chi^3(\mu,\nu)}\left(T\right)$ is a linear space.

Proof: It is trivial. Therefore omit the proof.

3.2. **Theorem.** Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of compact Bernstein polynomials of $\left(B_{mnk}\left(f,x\right)\right)$ of real numbers. Every open ball $D_{B_{mnk}(f,x)}\left(r,t\right)\left(T\right)$ is an open set in $B^{rI}_{\chi^3(\mu,\nu)}\left(T\right)$

Proof: Let $D_{B_{mnk}(f,x)}\left(r,t\right)\left(T\right)$ be an open ball with centre x and radius r with $D_{B_{mnk}(f,x)}\left(r,t\right)\left(T\right)=x\in\Lambda^3:$

$\mu\left(T\left(B_{mnk}\left(f,x\right)-T\left(B_{mnk}\left(f,y\right)\right),t\right)\right)\leq 1-(r+\epsilon)$ *or* $\nu\left(T\left(B_{mnk}\left(f,x\right)-T\left(B_{mnk}\left(f,y\right)\right),t\right)\right)\geq r+\epsilon\in rI.$ Let $y\in D^{c}_{B_{mnk}(f,x)}\left(r,t\right)\left(T\right).$ Then $\mu\left(T\left(B_{mnk}\left(f,x\right)-T\left(B_{mnk}\left(f,y\right)\right),t\right)\right)\leq 1-(r+\epsilon)$ and $\nu\left(T\left(B_{mnk}\left(f,x\right)-T\left(B_{mnk}\left(f,y\right)\right),t\right)\right)\geq r+\epsilon.$

Since $\mu\left(T\left(B_{mnk}\left(f,x\right)-T\left(B_{mnk}\left(f,y\right)\right),t\right)\right)\leq 1-(r+\epsilon)$ there exists $t_0\in(0,t)$ $\mu\left(T\left(B_{mnk}\left(f,x\right)-T\left(B_{mnk}\left(f,y\right)\right),t_0\right)\right)>1-(r+\epsilon)$ and $\nu\left(T\left(B_{mnk}\left(f,x\right)-T\left(B_{mnk}\left(f,y\right)\right),t_0\right)\right)<r+\epsilon.$ Putting $a_0=\mu\left(T\left(B_{mnk}\left(f,x\right)-T\left(B_{mnk}\left(f,y\right)\right),t_0\right)\right),$ we have $r_0>1-(r+\epsilon),$ there exists $s\in(0,1)$ such that $r_0>1-s>1-(r+\epsilon).$ For $r_0>1-s,$ we have $r_1,r_2\in(0,1)$ such that $r_0*r_1>1-s$ and $(1-r_0)\diamond(1-r_0)\leq s.$ Putting $r_3=max\{r_1,r_2\}.$ Consider the ball $D^{c}_{B_{mnk}(f,y)}\left(1-r_3,t-t_0\right)\left(T\right).$ We prove that $D^{c}_{B_{mnk}(f,y)}\left(1-r_3,t-t_0\right)\left(T\right)\subset D^{c}_{B_{mnk}(f,x)}\left(r,t\right)\left(T\right).$ Let $B_{mnk}\left(f,z\right)\in D^{c}_{B_{mnk}(f,y)}\left(1-r_3,t-t_0\right)\left(T\right),$ then $\mu\left(T\left(B_{mnk}\left(f,y\right)-T\left(B_{mnk}\left(f,z\right)\right),t-t_0\right)\right)>(r_3+\epsilon)$ and $\nu\left(T\left(B_{mnk}\left(f,y\right)-T\left(B_{mnk}\left(f,z\right)\right),t-t_0\right)\right)<1-(r_3+\epsilon).$ Therefore

$\mu\left(T\left(B_{mnk}\left(f,x\right)-T\left(B_{mnk}\left(f,z\right)\right),t\right)\right)\geq\mu\left(T\left(B_{mnk}\left(f,x\right)-T\left(B_{mnk}\left(f,y\right)\right),t_0\right)\right)*$
$\mu\left(T\left(B_{mnk}\left(f,y\right)-T\left(B_{mnk}\left(f,z\right)\right),t-t_0\right)\right)\geq(r_0*r_3)\geq(r_0*r_1)\geq(1-s)\geq(1-(r+\epsilon))$

and

$\nu\left(T\left(B_{mnk}\left(f,x\right)-T\left(B_{mnk}\left(f,z\right)\right),t\right)\right)\geq\nu\left(T\left(B_{mnk}\left(f,x\right)-T\left(B_{mnk}\left(f,y\right)\right),t_0\right)\right)\diamond$
$\mu\left(T\left(B_{mnk}\left(f,y\right)-T\left(B_{mnk}\left(f,z\right)\right),t-t_0\right)\right)\leq(1-(r_0+\epsilon))\diamond(1-(r_3+\epsilon))\leq(1-(r_0+\epsilon))\diamond$
$(1-(r_2+\epsilon))\leq s\leq r+\epsilon.$ Thus $B_{mnk}\left(f,z\right)\in D^{c}_{B_{mnk}(f,x)}\left(r,t\right)\left(T\right)$ and hence

$$D^{c}_{B_{mnk}(f,y)}\left(1-(r_3+\epsilon),t-t_0\right)\left(T\right)\subset D^{c}_{B_{mnk}(f,x)}\left(r,t\right)\left(T\right).$$

3.3. **Remark.** Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of compact Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers, $B^{rI}_{\chi^3(\mu,\nu)}(T)$ is an IRMS.

Proof: Define $\tau^{rI}_{\chi^3(\mu,\nu)}(T) =$
$\left\{A \subset B^{rI}_{\chi^3(\mu,\nu)}(T) \ for\ each B_{mnk}(f,x) \in A \exists t > 0\ and\ r \in (0,1)\ \ni:\ D_{B_{mnk}(f,x)}(r,t)(T)\right\}.$
Then $\tau^{rI}_{\chi^3(\mu,\nu)}(T)$ is a topology on $B^{rI}_{\chi^3(\mu,\nu)}(T)$.

3.4. **Remark.** The topology $\tau^{rI}_{\chi^3(\mu,\nu)}(T)$ on $B^{rI}_{\chi^3(\mu,\nu)}(T)$ is first countable.

3.5. **Theorem.** Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of compact Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers, $B^{rI}_{\chi^3(\mu,\nu)}(T)$ is a Housedorff space

Proof: Let $x, y \in B^{rI}_{\chi^3(\mu,\nu)}(T)$ such that $B_{mnk}(f,x) \neq B_{mnk}(f,y)$. Then $0 < \mu(T(B_{mnk}(f,x) - T(B_{mnk}(f,y)),t)) < 1$ and $0 < \nu(T(B_{mnk}(f,x) - T(B_{mnk}(f,y)),t)) < 1$. Putting $r_1 = \mu(T(B_{mnk}(f,x) - T(B_{mnk}(f,y)),t)), r_2 = \nu(T(B_{mnk}(f,x) - T(B_{mnk}(f,y)),t))$ and $r = max\{r_1+\epsilon, 1-(r_2+\epsilon)\}$. For each $r_0 \in (r+\epsilon, 1)$ there exists r_3 and r_4 such that $r_3 * r_4 \geq r_0 + \epsilon$ and $(1-(r_3+\epsilon)) \diamond (1-(r_4+\epsilon)) \leq (1-(r_0+\epsilon))$. Putting $r_5 = max\{r_3+\epsilon, (1-(r_4+\epsilon))\}$ and consider the open balls $D_{B_{mnk}(f,x)}\left(1-(r_5+\epsilon), \frac{t}{2}\right)$ and $D_{B_{mnk}(f,y)}\left(1-(r_5+\epsilon), \frac{t}{2}\right)$. Then clearly
$D^c_{B_{mnk}(f,x)}\left(1-(r_5+\epsilon), \frac{t}{2}\right) \bigcap D^c_{B_{mnk}(f,y)}\left(1-(r_5+\epsilon), \frac{t}{2}\right) = \phi$. For if there exists
$D^c_{B_{mnk}(f,x)}\left(1-(r_5+\epsilon), \frac{t}{2}\right) \bigcap D^c_{B_{mnk}(f,y)}\left(1-(r_5+\epsilon), \frac{t}{2}\right)$, then
$r_1 = \mu(T(B_{mnk}(f,x) - T(B_{mnk}(f,y)),t)) \geq$
$\mu\left(T\left(B_{mnk}(f,x) - T(B_{mnk}(f,z)), \frac{t}{2}\right)\right) * \mu\left(T\left(B_{mnk}(f,z) - T(B_{mnk}(f,y)), \frac{t}{2}\right)\right) \geq$
$(r_5+\epsilon) * (r_5+\epsilon) \geq (r_3+\epsilon) * (r_3+\epsilon) \geq r_0 + \epsilon > r_1 + \epsilon$ and
$r_2 = \nu(T(B_{mnk}(f,x) - T(B_{mnk}(f,y)),t)) \leq$
$\nu\left(T\left(B_{mnk}(f,x) - T(B_{mnk}(f,z)), \frac{t}{2}\right)\right) \diamond \nu\left(T\left(B_{mnk}(f,z) - T(B_{mnk}(f,y)), \frac{t}{2}\right)\right) \leq (1-(r_5+\epsilon)) \diamond$
$(1-(r_5+\epsilon)) \leq (1-(r_4+\epsilon)) \diamond (1-(r_4+\epsilon)) \leq (1-r_0+\epsilon > r_2+\epsilon)$ which is a contradiction. Hence $B^{rI}_{\chi^3(\mu,\nu)}(T)$ is a Housedorff space.

3.6. **Theorem.** Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of compact Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers, $B^{rI}_{\chi^3(\mu,\nu)}(T)$ is an IRMS and $\tau^{rI}_{\chi^3(\mu,\nu)}(T)$ is a topology on $B^{rI}_{\chi^3(\mu,\nu)}(T)$. Then a triple sequence of Bernstein polynomials $B_{mnk}(f,x) \in B^{rI}_{\chi^3(\mu,\nu)}(T), B_{mnk}(f,x) \longrightarrow f(x)$ if and only if $\mu(T(B_{mnk}(f,x) - T(B_{mnk}(f(x))),t)) \longrightarrow 1$ and
$\nu(T(B_{mnk}(f,x) - T(B_{mnk}(f(x))),t)) \longrightarrow 0$ as $m,n,k \in \infty$.

Proof: Fix $t_0 > 0$. Suppose $B_{mnk}(f,x) \longrightarrow f(x)$. Then for $r \in (0,1)$, there exists $c_0 \in \mathbb{N}^3$ such that $B_{mnk}(f,x) \in D_{B_{mnk}(f,x)}(r,t)(T)$ for all $m,n,k \geq c_0$,
$D_{B_{mnk}(f,x)}(r,t)(T) = \mu(T(B_{mnk}(f,x) - T(B_{mnk}(f(x))),t)) \leq 1-(r+\epsilon)$ *or*
$\nu(T(B_{mnk}(f,x) - T(B_{mnk}(f(x))),t)) \geq r+\epsilon \in I$, such that $D^c_{B_{mnk}(f,x)}(r,t)(T) \in F(I)$. Then $1-\mu(T(B_{mnk}(f,x) - T(B_{mnk}(f(x))),t)) < r+\epsilon$ and
$\nu(T(B_{mnk}(f,x) - T(B_{mnk}(f(x))),t)) < r+\epsilon$. Hence $\mu(T(B_{mnk}(f,x) - T(B_{mnk}(f(x))),t)) \longrightarrow 1$ and $\nu(T(B_{mnk}(f,x) - T(B_{mnk}(f(x))),t)) \longrightarrow 0$ as $m,n,k \to \infty$.
Conversely, if for each $t > 0$ $\mu(T(B_{mnk}(f,x) - T(B_{mnk}(f(x))),t)) \longrightarrow 1$ and $\nu(T(B_{mnk}(f,x) - T(B_{mnk}(f(x))),t)) \longrightarrow 0$ as $m,n,k \to \infty$, then for $r \in (0,1)$, there exists $c_0 \in \mathbb{N}^3$ such that $1-\mu(T(B_{mnk}(f,x) - T(B_{mnk}(f(x))),t)) < r+\epsilon$ and $\nu(T(B_{mnk}(f,x) - T(B_{mnk}(f(x))),t)) < r+\epsilon$, for all $m,n,k \geq c_0$. It follows that $\mu(T(B_{mnk}(f,x) - T(B_{mnk}(f(x))),t)) \leq 1-(r+\epsilon)$ and
$\nu(T(B_{mnk}(f,x) - T(B_{mnk}(f(x))),t)) \geq r+\epsilon$, for all $m,n,k \geq c_0$. Thus $B_{mnk}(f,x) \in D^c_{B_{mnk}(f,x)}(r,t)(T) \in F(I)$ for all $m,n,k \geq c_0$ and hence $B_{mnk}(f,x) \longrightarrow f(x)$.

4. Conclusions and Future Work

We introduced triple sequence spaces of compact Bernstein polynomials of rough $rI-$ convergence. For the reference sections, consider the following introduction described the main results are motivating the research.

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